Derivation of Black-Scholes PDE

November 5, 2013

1 Goal:

To show that under the model

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

the price $V(t, S_t)$ of a European-style derivative that pays $\phi(S_t)$ at time T satisfies

$$\begin{aligned} \frac{\partial}{\partial t}V(t,x) + \frac{\partial}{\partial x}V(t,x)rx + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,x)\sigma^2 x^2 - rV &= 0 \\ V(T,x) &= \phi(x). \end{aligned}$$

2 Ingredients

1. Ito's formula
2. Game-theory portfolio: We hold Δ_t shares of stock at time t and 1 share of V. We choose Δ_t such that the return of the portfolio is "deterministic".
3. No arbitrage principle: If a portfolio π_t satisfies

$$d\pi_t = \mu(t)\pi_t d_t, \tag{2.1}$$

then we must have $\mu(t) = r$, for all t.

3 Derivation of Black-Scholes PDE

1. Apply Ito's formula:

$$dV_t = \frac{\partial}{\partial t}V(t,S_t) + \frac{\partial}{\partial x}V(t,S_t)dS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2 dt.$$

Since

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

grouping the d_t and dB_t terms together we have

$$dV_t = \big[\frac{\partial}{\partial t}V(t,S_t) + \frac{\partial}{\partial x}V(t,S_t)rS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]dt + \big[\frac{\partial}{\partial x}V(t,S_t)\sigma S_t\big]dB_t. \quad (3.2)$$

2.

a. The game-theory portfolio π satisfies: $\pi_t = \Delta_t S_t + V_t$. By self-financing requirement:

$$\begin{aligned} d\pi_t &= \Delta_t dS_t + dV_t \\ &= \Delta_t\big(rS_t dt + \sigma S_t dB_t\big) + dV_t. \end{aligned}$$

Replace dV_t by (3.2) and group d_t, dB_t terms again we have

$$\begin{aligned} d\pi_t = \big[\Delta_t r S_t + \frac{\partial}{\partial t}V(t,S_t) + \frac{\partial}{\partial x}V(t,S_t)rS_t + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]dt \\ + \big[\Delta_t \sigma S_t + \frac{\partial}{\partial x}V(t,S_t)\sigma S_t\big]dB_t. \end{aligned}$$

b. We choose $\Delta_t = -\frac{\partial}{\partial x}V(t,S_t)$ to "kill" the dB_t term. Then

$$d\pi_t = \big[\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]dt. \quad (3.3)$$

3. To apply the no arbitrage principle, we need to rewrite the right hand side of (3.3) in the form of (2.1). We have

$$d\pi_t = \frac{\big[\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]}{\pi_t}\pi_t d_t. \quad (3.4)$$

This is in the form of (2.1) with

$$\mu(t) = \frac{\big[\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]}{\pi_t}.$$

Therefore, by the no arbitrage principle, we conclude that

$$\frac{\big[\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2\big]}{\pi_t} = r.$$

But $\pi_t = \Delta_t S_t + V_t = -\frac{\partial}{\partial x}V(t,S_t)S_t + V_t$. So we have

$$\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2 = r(-\frac{\partial}{\partial x}V(t,S_t)S_t + V_t).$$

In other words

$$\frac{\partial}{\partial t}V(t,S_t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(t,S_t)\sigma^2 S_t^2 + r\frac{\partial}{\partial x}V(t,S_t)S_t - rV_t = 0.$$

This is the Black-Scholes PDE.